Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic.

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joint work with K.Keimel, F. Montagna, W. Roth
Goal

The goal of this talk is to extend de Finetti’s theorem about coherent probability assignments to events of many-valued logic characterized by a condition of uncertainty.

- We axiomatically define imprecise probabilities (upper and lower probabilities) and we characterize them in terms of states.
- Is it possible to give an interpretation in terms of bets of imprecise probabilities?
- Can we prove a de Finetti-style coherence criterion for imprecise probabilities?
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Outline

1. Preliminaries

2. A Representation for Imprecise Probabilities
   - States and Imprecise probabilities over MV-algebras
   - Tools from functional analysis
   - Imprecise probabilities and sets of states

3. Coherent Books and Imprecise Probabilities
   - An interpretation of imprecise probabilities in terms of bets
   - An adequate coherence criterion
Definition

An **MV-algebra** is an algebra $A = (A, \oplus, \neg, 0)$ where:

- $(A, \oplus, 0)$ is a commutative monoid,
- $\neg$ is an involutive unary operation
- $1 = \neg 0$ is an element such that $a \oplus 1 = 1$,
- $a \rightarrow b = (\neg a) \oplus b$, is such that $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$.

In any MV-algebra $A$, we further define

$$a \odot b = \neg (\neg a \oplus \neg b), \quad a \ominus b = \neg (\neg a \oplus b)$$
Definition

A *valuation* on an MV-algebra $A$ is an homomorphism $\nu : A \rightarrow [0, 1]$.

Proposition

Let $X_A$ be the set of all valuations on the MV-algebra $A$. $X_A$ is a close subset of $[0, 1]^A$, thus an Hausdorff compact space with respect to the topology of pointwise convergence.
Remark

In a rational betting-game one will always identify events that have a priori the same value under every possible valuation, hence events will be the terms of the Lindenbaum-Tarski algebra of the Łukasiewicz logic, denoted by $\text{Fm}_L/\equiv_L$.

Proposition

$\text{Fm}_L/\equiv_L$ is an MV-algebra. Thus events may be regarded as elements of an MV-algebra.
Remark
For technical reasons, it is convenient to consider a wider class of events and to work in 2-divisible MV-algebras, i.e. MV-algebras in which we can multiply every element by $\frac{1}{2}$.

**Problem:** why can we do this?

- Every MV-algebra $A$ is embeddable in a 2-divisible one.
- Fuzzy events may be regarded as continuous functions from the Hausdorff compact space $X_A$ in $[0,1]$. Thus all dyadic numbers $q = \frac{m}{2^n}$ in the 2-divisible MV-algebra $A$ may be regarded as constant events, i.e. events that have a priori the same value under every possible valuation.
Remark

For technical reasons, it is convenient to consider a wider class of events and to work in 2-divisible MV-algebras, i.e. MV-algebras in which we can multiply every element by $\frac{1}{2}$.

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Definition

A *state* on the MV-algebra $\mathbf{A}$ is a map $P: \mathbf{A} \rightarrow [0, 1]$ such that for all $x, y \in \mathbf{A}$:

(i) $P(1) = 1$

(ii) If $x \odot y = 0$ then $P(x \oplus y) = P(x) + P(y)$

Theorem (Panti-Kroupa)

Let $\mathbf{A}$ an MV-algebra. There is a canonical bijective correspondence between the set of states on $\mathbf{A}$ and the set of regular probability measures on the Hausdorff compact space $X_{\mathbf{A}}$.

States play the same role for MV-algebras as probabilities for Boolean algebras.
A representation for imprecise probabilities

States and imprecise probabilities over MV-algebras

**Definition**

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**Theorem (Panti-Kroupa)**

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States play the same role for MV-algebras as probabilities for Boolean algebras.
Definition

An upper probability over a 2-divisible MV-algebra $A$ is a functional $P^+: A \rightarrow [0, 1]$ such that for all $x, y \in A$ and for all dyadic rational numbers $q = \frac{m}{2^n}$ in the unit interval:

(Ax1) is order preserving

(Ax2) $P^+(1) = 1$

(Ax3) $P^+(qx) = qP^+(x)$

(Ax4) If $x \odot y = 0$ then $P^+(x \oplus y) \leq P^+(x) + P^+(y)$

(Ax5) If $q \odot x = 0$ allora $P^+(q \oplus x) = q + P^+(x)$

Definition

A lower probability over a 2-divisible MV-algebra $A$ is a functional $P^-: A \rightarrow [0, 1]$ with the same properties of $P^+$ except (Ax4) that becomes:

(Ax4)' If $x \odot y = 0$ then $P^-(x \oplus y) \geq P^-(x) + P^-(y)$
**Definition**

An *upper probability* over a 2-divisible MV-algebra $A$ is a functional $P^+: A \rightarrow [0, 1]$ such that for all $x, y \in A$ and for all dyadic rational numbers $q = \frac{m}{2^n}$ in the unit interval:

1. **(Ax1)** is order preserving
2. **(Ax2)** $P^+(1) = 1$
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**Definition**

A *lower probability* over a 2-divisible MV-algebra $A$ is a functional $P^-: A \rightarrow [0, 1]$ with the same properties of $P^+$ except (Ax4) that becomes:

**(Ax4)'** If $x \odot y = 0$ then $P^-(x \oplus y) \geq P^-(x) + P^-(y)$
Definition

Let $X$ an Hausdorff compact space. A functional $u : C(X) \to \mathbb{R}$ is:

- **order preserving** if $f \leq g$ implies $u(f) \leq u(g)$
- **normalized** if $u(1) = 1$
- **subadditive** (resp. **superadditive**) if $u(f + g) \leq u(f) + u(g)$ (resp. $\geq$)
- **homogeneous** if $u(rf) = ru(f)$ for all $f \in C(X)$ and all $r \geq 0$,
- **sublinear** (resp. **superlinear**) if $u$ is homogeneous and subadditive (resp. superadditive)
- **linear** if $u$ is sublinear and superlinear.
- **weakly linear** if $u(f + r \cdot 1) = u(f) + r$ for all $r \in \mathbb{R}$ and for all $f \in C(X)$. 

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Theorem

Let $A$ be an MV-algebra.

There is a canonical bijection between states on $A$ and order preversing, normalized, homogeneous and linear functionals on $C(X_A)$.

If $A$ is a 2-divisible MV-algebra

There is a canonical bijection between upper (resp. lower) probabilities on $A$ and order preversing, normalized, homogeneous, weakly linear and sublinear (resp. superlinear) functionals on $C(X_A)$.

Remark

In the 2-divisible case, we are in the position to apply classical tools of functional analysis to deal with states and imprecise probabilities on MV-algebras thanks to their counterparts in the dual of $C(X)$, where $X$ is an Hausdorff compact space.
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Let $X$ an Hausdorff compact space and $C(X)^*$ the dual vector space of $C(X)$. We endow $C(X)^*$ with the *weak* topology.

**Definition**

The *weak* topology is the weakest topology such that the evaluation maps $\mu \mapsto \mu(f) : C(X)^* \to \mathbb{R}$ are continuous for all $f \in C(X)$.

**Proposition**

Let $\mathcal{P}(X)$ the set of all order preserving, normalized and linear functionals on $C(X)$. $\mathcal{P}(X)$ is a convex subset of $\mathcal{M}(X)$ which is compact in the *weak* topology.
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Proposition

For every weak* -closed subset $K$ of $\mathcal{P}(X)$, the functional $K^+: C(X) \to \mathbb{R}$ defined by $K^+(f) = \max_{\mu \in K} \mu(f)$ is well defined.

Theorem

- $K^+(f) = \max_{\mu \in K} \mu(f)$ is an order preserving, normalized, sublinear and weakly linear functional.
- Conversely, for every order preserving, normalized, sublinear and weakly linear functional $u: C(X) \to \mathbb{R}$, $u_{\leq} = \{\mu \in \mathcal{P}(X) : \mu \leq u\}$ is a weak* -closed convex subset of $\mathcal{P}(X)$. 
Proposition

For every weak**-closed subset \( K \) of \( \mathcal{P}(X) \), the functional \( K^+: C(X) \to \mathbb{R} \) defined by \( K^+(f) = \max_{\mu \in K} \mu(f) \) is well defined.

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- \( K^+(f) = \max_{\mu \in K} \mu(f) \) is an order preserving, normalized, sublinear and weakly linear functional.
- Conversely, for every order preserving, normalized, sublinear and weakly linear functional \( u: C(X) \to \mathbb{R} \), \( u_\leq = \{ \mu \in \mathcal{P}(X) : \mu \leq u \} \) is a weak**-closed convex subset of \( \mathcal{P}(X) \).
Theorem

- For every weak*-closed subset $K$ of $\mathcal{P}(X)$ the functional defined by $K^{-}(f) = \min_{\mu \in K} \mu(f)$ is an order preserving, normalized, superlinear and weakly linear functional.
- Conversely, for every order preserving, normalized, superlinear and weakly linear functional $u : C(X) \to \mathbb{R}$, $u_{\geq} = \{ \mu \in \mathcal{P}(X) : \mu(f) \geq u(f) \text{ for all } f \in C(X) \}$ is a nonempty weak* closed convex subset of $\mathcal{P}(X)$.

Theorem

The maps $K \mapsto (K^+, K^{-})$ and $u \mapsto (u_{\leq}, u_{\geq})$ are mutually inverse bijections.
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Theorem

Let $\mathbf{A}$ a 2-divisible MV-algebra

- There is a canonical bijection between upper (resp. lower) probabilities on $\mathbf{A}$ and order preversing, normalized, homogeneous, sublinear (resp. superlinear) and weakly linear functionals on $C(X_{\mathbf{A}})$.

- There is a canonical bijection between states on $\mathbf{A}$ and order preversing, normalized, homogeneous and linear functionals on $C(X_{\mathbf{A}})$.
A Representation for Imprecise Probabilities

Imprecise probabilities and sets of states

Theorem (Representation Theorem for Upper and Lower Probabilities)

- An upper probability is the maximum of a non-empty convex set of states, close with respect to the weak* topology and vice versa.
- A lower probability is the minimum of a non-empty convex set of states, close with respect to the weak* topology and vice versa.

Remark

Two sets $A$ and $B$ of states on an MV-algebra $A$ represent the same imprecise probability if and only if they have the same closed convex hull.
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Example

Let us suppose to know that the extraction of the bingo are biased. If we don’t know how it is biased, the event “A great number will be drawn out” is a fuzzy event under particularly severe uncertainty.

In this case, a bookmaker may not be willing to quantify with a number his subjective opinion that the event $\varphi$ will happen. He will probably prefer associating to $\varphi$ an interval of probability, choosing a betting odd $\beta$ for betting on $\varphi$ and a betting odd $\alpha \leq \beta$ for betting against $\varphi$. 
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Definition

A book that contains fuzzy events under particularly severe uncertainty is a finite set of the form $\Delta = \{ (\varphi_1, [\alpha_1, \beta_1]), \ldots, (\varphi_n, [\alpha_n, \beta_n]) \}$, where for $0 \leq \alpha_i \leq \beta_i \leq 1$.

- If $\alpha_i = \beta_i$ for all $i = 1, \ldots, n$ we are again in the case of reversible betting game.
- If $\alpha_i < \beta_i$ the conditions are disadvantageous for the bettor:
  - if the bettor bets on $\phi$ he can win at the most $\lambda(1 - \beta)$ and he can loose up to $\lambda\beta$.
  - if the bettor bets against $\phi$ he can win at the most $\lambda\alpha$ and he can loose up to $\lambda(1 - \alpha)$.

Remark

If the bookmaker is a rational being, he wants to protect himself from the risk to loose a huge amount of money. Thus he forbids negative bets because he doesn’t want to reverse his role with the bettor.
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Remark

If the bookmaker is a rational being, he wants to protect himself from the risk to loose a huge amount of money. Thus he forbids negative bets because he doesn’t want to reverse his role with the bettor.
In order to give the rules for betting in a non-reversible betting game, firstly we replace every element \((\varphi, [\alpha, \beta])\) of the book with the pairs \((\varphi, \beta), (\neg \varphi, 1 - \alpha)\).

- If a bettor wants to bet on \(\varphi\), he pays \(\lambda \beta\) with \(\lambda > 0\) and he will receive \(\lambda \nu(\varphi)\).
- If he wants to bet against \(\varphi\), he bets on \(\neg \varphi\); hence he pays \(\lambda (1 - \alpha)\) with \(\lambda > 0\) and he will receive \(\lambda \nu(\neg \varphi)\)

where \(\nu\) is a valuation in \([0, 1]\).

**Definition (Subjective Approach)**

- The upper probability of \(\varphi\) is the number \(\beta\) that a rational and non-reversible bookmaker would propose for the preceding bet.
- The lower probability of \(\varphi\) is instead the number \(\alpha\).
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Definition

Let $\Delta$ a fixed book.

- A **winning strategy** (resp. a **loosing strategy**) for the bettor based on $\Delta$ consists of a system of bets on a finite subset of events in $\Delta$ which ensures to the bettor a strictly positive (resp. strictly negative) payoff independently of the valuation.

- A **bad bet** (resp. a **good bet**) is a bet on an element of $\Delta$ for which we can find a system of bets on a finite subset of events in $\Delta$ which ensures to the bettor a better payoff (resp. a worse payoff) independently of the valuation.
Coherence Criterion (Reversible betting-game)

A book $\Delta$ is said to be *rational* or *coherent* if there is no *winning strategy* for the bettor.

Theorem

Let $\Delta$ be a book in a reversible game. The following are equivalent:

- There is no winning strategy for the bettor.
- There is no loosing strategy for the bettor.
- There is no bad bet for the bettor.
- There is no good bet for the bettor.
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Coherent Books and Imprecise Probabilities

An adequate coherence criterion

Example

Let $\Gamma = \{ (\varphi, \frac{1}{3}), (\neg \varphi, 1), (\psi, \frac{1}{3}), (\neg \psi, 1), (\varphi \lor \psi, 1), (\neg (\varphi \lor \psi), 1) \}$

- The bettor has no winning strategy for this book because if both $\varphi$ and $\psi$ are false, then he cannot win anything.
- This book does not look rational: the bookmaker might make it more attractive for the bettor by reducing the betting odd on $\varphi \lor \psi$ to $\frac{2}{3}$, without any loss of money if the bettor plays his best strategy.

Example

Let $\Delta$ a book that contains $(\varphi, \alpha)$, $(\neg \varphi, \beta)$ with $\alpha + \beta > 1$.

- The bettor has a loosing strategy, namely, betting 1 on both $\varphi$ and $\neg \varphi$. He has also a good bet, namely, betting nothing.
- This book cannot be considered irrational: it can be obtained from an element $(\varphi, [.30, .60])$ that is obviously acceptable.
Example

Let \( \Gamma = \{ (\varphi, \frac{1}{3}), (\neg\varphi, 1), (\psi, \frac{1}{3}), (\neg\psi, 1), (\varphi \lor \psi, 1), (\neg(\varphi \lor \psi), 1) \} \)

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- This book cannot be considered irrational: it can be obtained from an element \( (\varphi, [.30, .60]) \) that is obviously acceptable.
In the case of imprecise probabilities and non-reversible games,

**Remark**
- The non existence of a winning strategy for the bettor is a necessary but not sufficient condition for rationality.
- The non existence of a loosing strategy, or the non existence of a good bet are too strong criteria, in the sense that it is not reasonable to expect them to hold.

**Coherence Criterion (Non-reversible betting-game)**

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**Coherence Criterion (Non-reversible betting-game)**

A book $\Delta$ is said to be *rational* or *coherent* if there is no *bad bet* for the bettor.
Theorem (Fedel, Keimel, Montagna, Roth)

Let $\Delta$ be a book in a non-reversible game over a 2-divisible MV-algebra $A$ of events. Then the following are equivalent:

1. There is no bad bet for the bettor based on $\Delta$.
2. There is an upper probability $P^+$ over the MV-algebra $A$ of events such that, if $(\varphi, \alpha) \in \Delta$, then $P^+(\varphi) = \alpha$. 
Theorem (Fedel, Keimel, Montagna, Roth)

Let $\Delta$ be a book in a non-reversible game over a 2-divisible MV-algebra $A$ of events. Then the following are equivalent:

- There is no good bet for the bettor based on $\Delta$.
- There is a lower probability $P^-$ over the MV-algebra $A$ of events such that, if $(\varphi, \alpha) \in \Delta$, then $P^-(\varphi) = \alpha$.

Problem: Why does the non-existence of a good bet, which is not a rationality criterion, correspond to a significant property, that is, extendability to a lower probability?

Remark

If betting $\lambda > 0$ on $\varphi$ is a good bet in a game in which only positive bets are allowed, then betting $-\lambda$ on $\varphi$ is a bad bet in a game in which only negative bets are allowed. Hence, accordance with a lower probability corresponds to a reasonable rationality criterion.
Theorem (Fedel,Keimel,Montagna,Roth)

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Let $\Delta$ be a book in a non-reversible game over a 2-divisible MV-algebra $A$ of events. Then the following are equivalent:

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2. There is a lower probability $P^-$ over the MV-algebra $A$ of events such that, if $(\varphi, \alpha) \in \Delta$, then $P^-(\varphi) = \alpha$.

**Problem:** Why does the non-existence of a good bet, which is not a rationality criterion, correspond to a significant property, that is, extendability to a lower probability?

**Remark**

If betting $\lambda > 0$ on $\varphi$ is a good bet in a game in which only positive bets are allowed, then betting $-\lambda$ on $\varphi$ is a bad bet in a game in which only negative bets are allowed. Hence, accordance with a lower probability corresponds to a reasonable rationality criterion.
Theorem (Fedel, Keimel, Montagna, Roth)

Let $\Delta$ be a book in a non-reversible game over a 2-divisible MV-algebra $A$ of events. Then the following are equivalent:

- There is no good bet and no bad bet based on $\Delta$.
- There is a state $P$ on $A$ such that, if $(\varphi, \alpha) \in \Delta$, then $P(\varphi) = \alpha$. 
...Thanks!